Vector Calculus

Vector Differential operator ∇ (del / nabla)

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

where i, j, k are unit vectors.

Gradient

The gradient of a scalar function $f(x, y, z)$ is defined as

$$\nabla f = \text{grad } f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

Note: Gradient is the operation of del on a scalar point function and it generates a vector point function.

Examples

(1) Find grad f, for $f = 3x^2y - y^3z^2$ at point $(1, -2, -1)$

Soln: Here $f = 3x^2y - y^3z^2$

$$\text{grad } f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\text{grad } f = i\frac{\partial(3x^2y - y^3z^2)}{\partial x} + j\frac{\partial(3x^2y - y^3z^2)}{\partial y} + k\frac{\partial(3x^2y - y^3z^2)}{\partial z}$$

$$\text{grad } f = i(6xy) + j(3x^2 - 3y^2z^2) + k(-2y^3z)$$

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Now

$$\text{grad } f = i\,[6(1)(-2)] + j[3(1)^2 - 3(-2)^2 + (-1)^3] + k[-2(-2)^3(-1)]$$

$$\text{grad } f \Big|_{(1,-2,-1)} = -12i - 9j - 16k$$

(2) Find gradient of

$f(x,y,z) = 2x^3 - 3(x^2+y^2)z + \tan^{-1}(xz)$ at $(1,1,1)$

Soln:

$$\nabla f = \text{grad } f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\Rightarrow \nabla f = i\left(-6xz + \frac{1}{1+x^2z^2}\cdot z\right) + j(-6yz) + k\left(6z^2 - 3(x^2+y^2) + \frac{1}{1+x^2z^2}\cdot x\right)$$

At $(1,1,1)$, grad f is

$$\text{grad } f \Big|_{(1,1,1)} = \nabla f \Big|_{(1,1,1)} = -\frac{11}{2}i - 6j + \frac{1}{2}k$$

(3) Find $\nabla\phi$ for $\phi = x^2 \sin 2y$ at point $(1, \pi/2)$

Soln:

$$\nabla\phi = \text{grad } \phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

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$$\Rightarrow \nabla\phi = i\frac{\partial}{\partial x}(x^2\sin 2y) + j\frac{\partial}{\partial y}(x^2\sin 2y) + k\frac{\partial}{\partial z}(x^2\sin 2y)$$

$$= i(2x\cdot\sin 2y) + j(x^2\cdot 2\cos 2y) + k(0)$$

$$= (2x\sin 2y)i + (2x^2\cos 2y)j$$

$$\nabla\phi_{(1,\pi/2)} = (2(1)\sin\pi)i + (2(1)^2\cos\pi)j$$

$$= 0i - 2j$$

$$= -2j$$

Exercise:

(1) Find grad ϕ for $\phi = 2x^3y^2z^4$ at $(1,2,3)$

(2) Find grad f for $f = 2x^2 + 3y^2 + z^2$ at $P(2,1,3)$

(3) Find ∇f for $f = 3e^x\cos yz$

(4) Find grad v for $v = x^3y^3z^3$ at point $(-1,-1,-1)$

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