# Title : Time Varying Fields and Maxwell's Equations

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Lecture No : (02)

Source of information : Field Theory by Dr. Sandeep Wali, Page no. 9-1 to 9-17

# Time-Varying Fields

Stationary charges → electrostatic fields

Steady currents → magnetostatic fields

Time-varying currents → electromagnetic fields

Only in a non-time-varying case can electric and magnetic fields be considered as independent of each other. In a time-varying (dynamic) case the two fields are interdependent. A changing magnetic field induces an electric field, and vice versa.

## The Continuity Equation

Electric charges may not be created or destroyed (the principle of conservation of charge).

Consider an arbitrary volume *V* bounded by surface *S*. A net charge *Q* exists within this region. If a net current *I* flows across the surface out of this region, the charge in the volume must decrease at a rate that equals the current:

$\rho_V$

$$I = \int_S \overline{J} \cdot \overline{dS} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho_V dv$$

Divergence theorem

$$\int_V \nabla \cdot \overline{J} dv = -\int_V \frac{\partial \rho_V}{\partial t} dv$$

Partial derivative because may be a function of both time and space

This equation must hold regardless of the choice of *V*, therefore the integrands must be equal:

$$\nabla \cdot \overline{J} = -\frac{\partial \rho_V}{\partial t} \quad (A/m^3)$$

the equation of continuity

For steady currents

$$\nabla \cdot \overline{J} = 0$$

Kirchhoff's current law

follows from this

that is, steady electric currents are divergences or solenoidal.

## Displacement Current

For magnetostatic field, we recall that

$$\nabla \times \overline{H} = \overline{J}$$

Taking the divergence of this equation we have

$$\nabla \cdot \left(\nabla \times \overline{H}\right) = 0 = \nabla \cdot \overline{J}$$

However the continuity equation requires that

$$\nabla \cdot \overline{J} = -\frac{\partial \rho_V}{\partial t} \neq 0$$

Thus we must modify the magnetostatic curl equation to agree with the continuity equation. Let us add a term to the former so that it becomes

$$\nabla \times \overline{H} = \overline{J} + \overline{J}_d$$

where $\overline{J}$ is the conduction current density $\overline{J} = \sigma_E \overline{E}$ , and $\overline{J}_d$ is to be determined and defined.

## Displacement Current continued

Taking the divergence we have

$$\nabla \cdot \left(\nabla \times \overline{H}\right) = 0 = \nabla \cdot \overline{J} + \nabla \cdot \overline{J}_d \longrightarrow \nabla \cdot \overline{J}_d = -\nabla \cdot \overline{J}$$

In order for this equation to agree with the continuity equation,

Gauss' law

$$\nabla \cdot \overline{J}_d = -\nabla \cdot \overline{J} = \frac{\partial \rho_V}{\partial t} = \frac{\partial}{\partial t}\left(\nabla \cdot \overline{D}\right) = \nabla \cdot \frac{\partial \overline{D}}{\partial t}$$

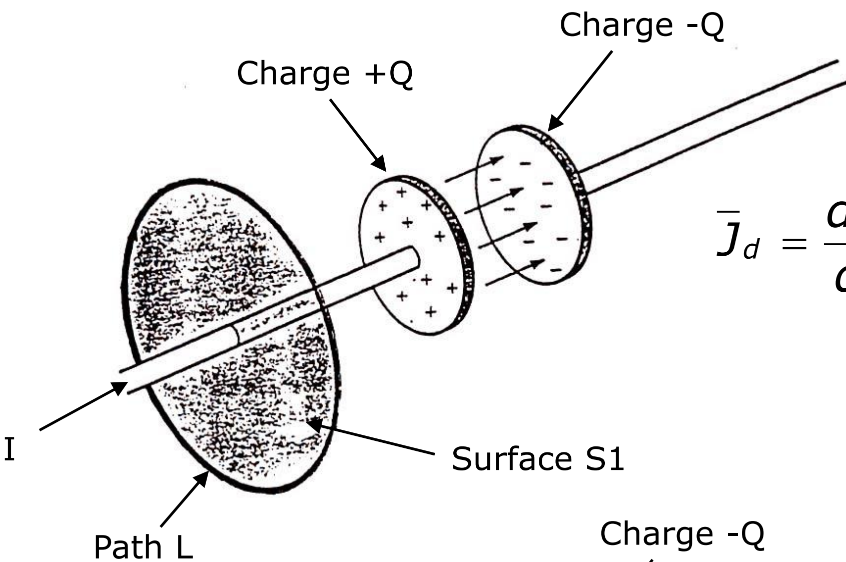
$$\overline{J}_d = \frac{\partial \overline{D}}{\partial t} \longleftarrow \text{displacement current density}$$

$$\boxed{\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}}$$

$$\int_S \nabla x \overline{H} \cdot \overline{ds} = \int_S \left(\overline{J} + \frac{\partial \overline{D}}{\partial t}\right) \cdot \overline{ds} \longrightarrow \boxed{\oint_L \overline{H} \cdot \overline{dl} = I + \int_S \frac{\partial \overline{D}}{\partial t} \cdot \overline{ds}}$$

Stokes' theorem

# Displacement Current continued




$$\bar{J}_d = \frac{d\bar{D}}{dt}$$

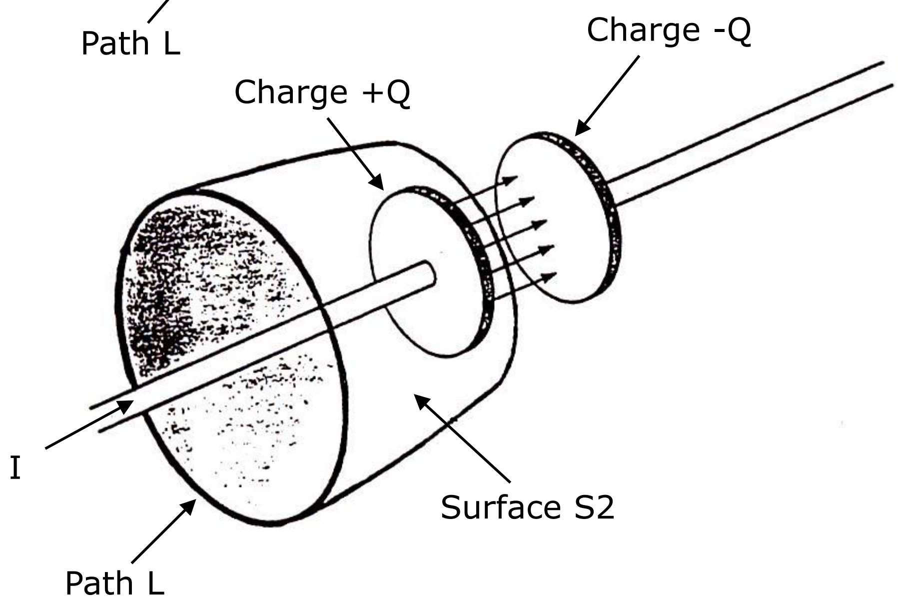
The total current density is $\bar{J} + \bar{J}_d$. In the first equation $\bar{J}_d = 0$ so it remains valid. In the second equation $\bar{J} = 0$ so that

$$\oint_L \bar{H} \cdot d\bar{l} = \int_{S_2} \bar{J}_d \cdot d\bar{S} = \frac{d}{dt} \underbrace{\int_{S_2} \bar{D} \cdot d\bar{S}}_{Q} \leftarrow \boxed{\bar{J} = 0}$$

$$= \frac{dQ}{dt} = I$$

$$\oint_{S_1+S_2} \bar{D} \cdot d\bar{S} = Q$$

$$\int_{S_1} \bar{D} \cdot d\bar{S} = 0$$




So we obtain the same current for either surface though it is conduction current in $S_1$ and displacement current in $S_2$.

# Faraday's Law

Faraday discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In differential (or point) form this experimental fact is described by the following equation

$$\nabla x \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

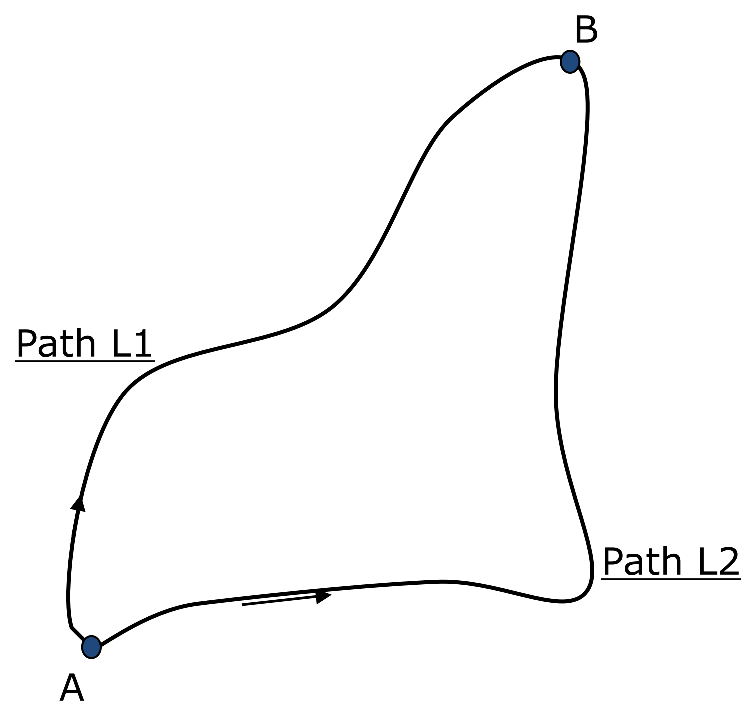
Taking the surface integral of both sides over an open surface and applying Stokes' theorem, we obtain

Integral form ⟶ $\oint_L \overline{E} \cdot \overline{dl} = -\frac{\partial}{\partial t}\int_S \overline{B} \cdot \overline{dS} = -\frac{\partial \psi}{\partial t}$

where $\psi$ is the magnetic flux through the surface S.

# Faraday's Law continued

## Time-varying electric field is not conservative.




The effect of electromagnetic induction. When time-varying magnetic fields are present, the value of the line integral of $\overline{E}$ from A to B may depend on the path one chooses.

Suppose that there is only one unique voltage $V_{AB} = V_A - V_B$. Then

$$V_{AB} = \oint_{L_1} \overline{E} \cdot d\overline{l} = \oint_{L_2} \overline{E} \cdot d\overline{l}$$

However,

$$\oint_L \overline{E} \cdot \overline{dl} = \underbrace{\int_{L_1} \overline{E} \cdot \overline{dl}}_{V_{AB}} - \underbrace{\int_{L_2} \overline{E} \cdot \overline{dl}}_{V_{AB}} = -\frac{\partial \psi}{\partial t}$$

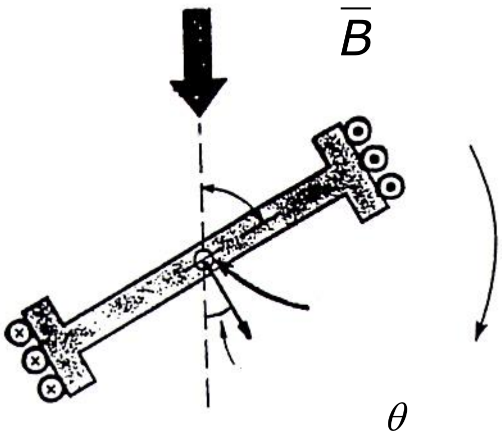
$$V_{AB} - V_{AB} \neq 0$$

Thus $V_{AB}$ can be unambiguously defined only if $\partial \psi / \partial t = 0$ .(in practice, if $\lambda >>$ than the dimensions of system in question)

# Faraday's Law continued

Example: An $N$-turn coil having an area $A$ rotates in a uniform magnetic field $\overline{B}$. The speed of rotation is $n$ revolutions per second. Find the voltage at the coil terminals.

$\omega_M = 2\pi n$




$$\psi = \int_S \overline{B} \cdot d\overline{S}$$

$$= BA\sin\theta$$

$$\theta = 2\pi n t \ \text{rad}$$

Rotation

$$v(t) = -N\frac{\partial \psi}{\partial t}$$

Shaft

$$d\overline{S} = dA\overline{e}_n$$

$$(90^o - \theta)$$

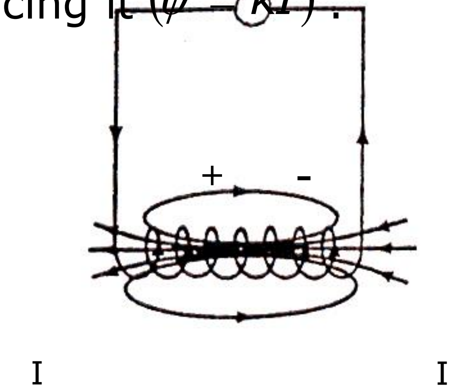
$$= -N\frac{\partial}{\partial t}\left[BA\sin(2\pi n t)\right]$$

$$\cos(90^\circ - \theta) = \sin\theta$$

$$= -\underbrace{2\pi n NBA}_{amplitude}\cos(\underbrace{2\pi n}_{\omega}t)$$

# Inductance

A circuit carrying current $I$ produces a magnetic field $\overline{B}$ which causes a flux $\psi = \int \overline{B} \cdot d\overline{S}$ to pass through each turn of the circuit. If the medium surrounding the circuit is linear, the flux $\psi$ is proportional to the current $I$ producing it $(\psi = kI)$.




Magnetic field **B** produced by a circuit.

For a time-varying current, according to Faraday's law, we have

$$V = N\frac{\partial \psi}{\partial t}$$ (voltage induced across coil)

$$= Nk\frac{\partial I}{\partial t} = L\frac{\partial I}{\partial t}$$

$$L = Nk = \frac{N\psi}{I} = \frac{\lambda}{I}$$ , H (henry)

Self-inductance $L$ is defined as the ratio of the magnetic flux linkage $\lambda$ to the current $I$.

# Inductance continued

$$\nabla x \overline{E} = -\frac{\partial \overline{B}}{\partial t} \longrightarrow \oint_L \overline{E} \cdot \overline{dl} = -\frac{\partial}{\partial t}\int_S \overline{B} \cdot d\overline{S}$$

Resistance

Since $R=0$, $\overline{E}$ has only radial component and therefore the segments 3 and 4 contribute nothing to the line integral.

$$\underbrace{\int_1 \overline{E} \cdot \overline{dl} + \int_2 \overline{E} \cdot \overline{dl}}_{v(z)-v(z+h)} = -\mu \frac{\partial}{\partial t}\int_S \overline{H} \cdot d\overline{S} = +\mu \frac{\partial}{\partial t}\int_\rho \left(H_\phi \overline{e}_\phi\right) \cdot \left(h d\rho \overline{e}_\phi\right)$$

(with $H_\phi = \frac{I}{2\pi\rho}$)

$$= \frac{\mu h}{2\pi}\frac{\partial I}{\partial t}\int_a^b \frac{1}{\rho} d\rho = \frac{\mu h}{2\pi} \ln\left(\frac{b}{a}\right)\frac{\partial I}{\partial t}$$

$$L\frac{dI(z)}{dt} = -\frac{v(z+h)-v(z)}{h}$$

(from previous work)

$$\boxed{L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)} \quad \text{H/m}$$

- Assignment:
- 1) write a short note on continuity equation of Current
- 2) Derive the relation between I and J?
- Write these questions in the workbook given to you and Submit it by 31/03/2020
- For any Query pl Contact: Ms Richa Dubey (9998037615) richa.dubey@srict.in