Shroff S.R. Rotary Institute of Chemical Technology (SRICT)

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Response of First Order Systems

First Order System – Differential Equations

- The dynamic control configuration of a system is described by means of differential equations.
- The order of differential equation representing dynamics of the system is same as that of order of control system.
- Behavior of first order system can be represented by means of first order differential equation.
- To study how the system behaves (response of system), it is necessary to develop transfer function of system.
- Transfer function relates output of the process to input of the process and relationship is given by block diagram

$$\boldsymbol{G(s)=\frac{Y(s)}{X(s)}=\frac{Laplace\ transform\ of\ output\ (deviation\ variable)}{Laplace\ transform\ of\ input(deviation\ variable)}}$$

  - Transfer function is necessary for development of input output model
  - Describes dynamic behavior of output for any change in input variable

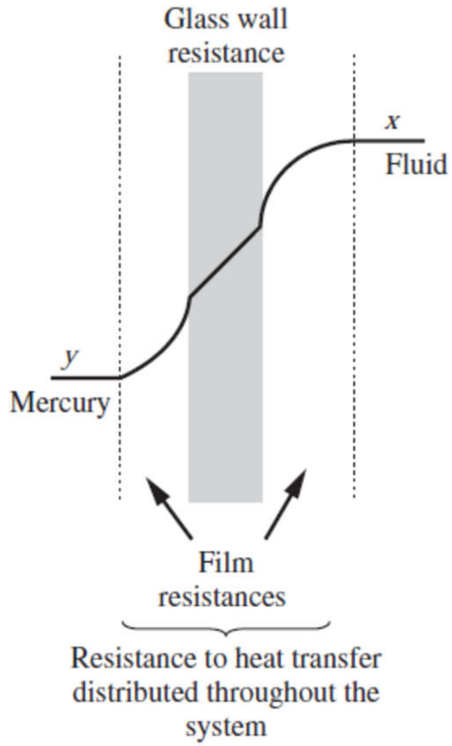
Transfer Functions

- Transfer function of first order system
  - Considering the unsteady state behavior of ordinary mercury in glass thermometer

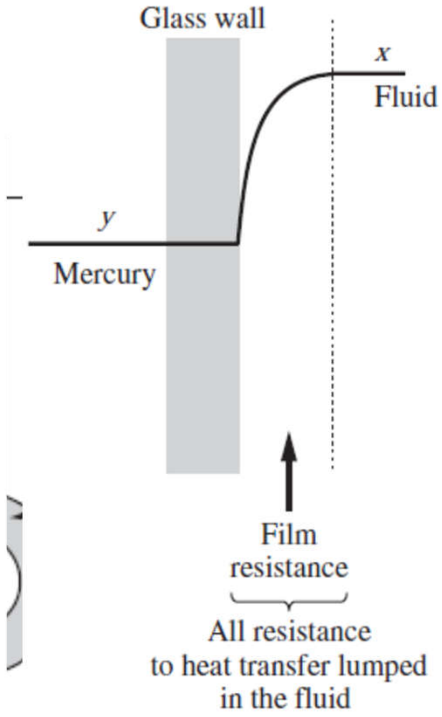
- Aim – Calculate the response of thermometer for change in fluid temperature
- Assumptions

1. All resistance to heat transfer lies in the film surrounding the bulb. (Resistance offered by glass and mercury is neglected)
2. All thermal capacity is within mercury. Mercury assumes a uniform temperature through out
3. Glass wall containing mercury does not expand or contract during transient response.

Initial steady state for thermometer – before time t = 0, there is no change in temperature with time



Actual temperature profile



Lumped temperature profile

Applying an unsteady state energy balance

Heat input rate - Heat output rate = Rate of accumulation

$$hA(x-y)-0=mC\frac{dy}{dt} \quad \text{Equation 1}$$

$$A = surface\ area\ of\ bulb\ for\ heat\ transfer, m^2$$
$$C = heat\ capacity\ of\ mercury, \frac{J}{kg\ K}$$

$$m = mass\ of\ mercury\ in\ bulb\ , kg$$

$$t = time, sec$$

$$h = film\ heat\ transfer\ coefficient\ , \frac{W}{m^2\ K}$$

Rate of heat flow through the film surrounding the bulb causes rise in internal energy of mercury.

This increase is manifested as expansion of mercury through the graduated mercury column indicating temperature

Prior to application of change, i.e. thermometer at steady state the derivative dy/dt is zero.

$$hA(x_s-y_s)=0; for\ t<0 \quad \text{Equation 2}$$

Equation 1 - Equation 2

$$hA[(x-x_s)-(y-y_s)]=mC\frac{d(y-y_s)}{dt}$$

Introducing deviation variables

$$X=x-x_s \qquad Y=y-y_s$$

$$hA[X-Y]=mC\frac{dY}{dt}$$

$$[X-Y]=\tau\frac{dY}{dt} \qquad \tau=\frac{mC}{hA}$$

$$\tau\, sY(s)-Y(0)=X(s)-Y(s)$$

$$Since\ Y(0) = 0$$

$$\tau\, sY(s) = X(s) - Y(s)$$

$$(\tau\, s + 1)Y(s) = X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{(\tau s + 1)}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{(\tau s + 1)} = \frac{Laplace\ transform\ of\ output}{Laplace\ transform\ of\ input}$$

Any system represented in such a form is called as first order system.

To determine the transfer function of process

1. Write the appropriate balance equations
2. Represent the balance equations in deviation form
3. Apply Lapalce transform to balance equation
4. Represent the equation in form of ratio of output to that of input.

Standard form for First order Transfer Functions

General form of first order system is

$$\tau \frac{dy}{dt} + y = K_p\, x(t)$$

The initial conditions are

$$y(0) = y_s = K_p\, x(0) = K_p x_s$$

Introducing deviation variables

$$X = x - x_s \qquad Y = y - y_s$$

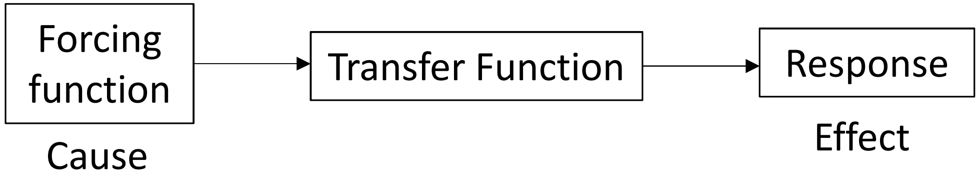
$$\tau \frac{dY}{dt} + \mathrm{Y} = K_p\, X(t)$$

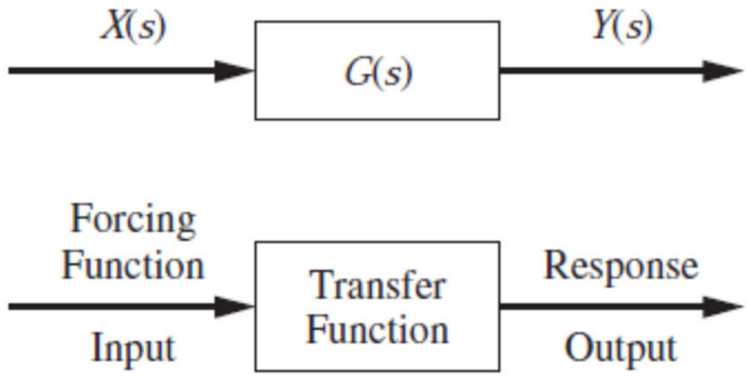
$$\tau\, sY(s) + Y(s) = K_p X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{K_p}{(\tau s + 1)} \qquad K_p = steady\ state\ gain$$

The steady state gain represent steady state value attained by system once it is disturbed by step input.

Properties of transfer function





Transfer Function= $G(s) = \frac{Y(s)}{X(s)}$

$G(s) = transfer\ function$

$X(s) = Laplace\ transform\ of\ input\ in\ deviation\ form$

$Y(s) = Laplace\ transform\ of\ output\ in\ deviation\ form$

$$Y(s) = G(s)X(s)$$

Principle of superposition is applicable to transfer function

$$If\ X(s) = a_1X_1(s) + a_2X_2(s)$$

$$then\ Y(s) = G(s)X(s)$$

$$= G(s)[a_1X_1(s) + a_2X_2(s)]$$

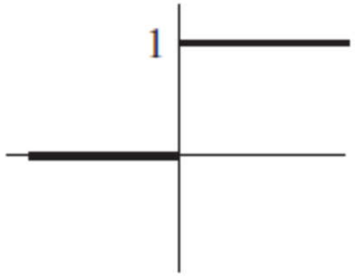
$$= G(s)a_1X_1(s) + G(s)a_2X_2(s) \quad = a_1\ Y_1(s) + a_2Y_2(s)$$

Transient response of first order system

- Response of first order system different kinds of inputs.

1. Step Function

Mathematically represented as



$$\boldsymbol{X(t)} = \begin{Bmatrix} \mathbf{0} & \boldsymbol{for\ \ t<0} \\ \mathbf{1} & \boldsymbol{for\ \ t>0} \end{Bmatrix}$$

For step change of magnitude A

$$\boldsymbol{X(t) = A} \qquad \boldsymbol{X(s) = \frac{A}{s}}$$

Transfer function for first order system is

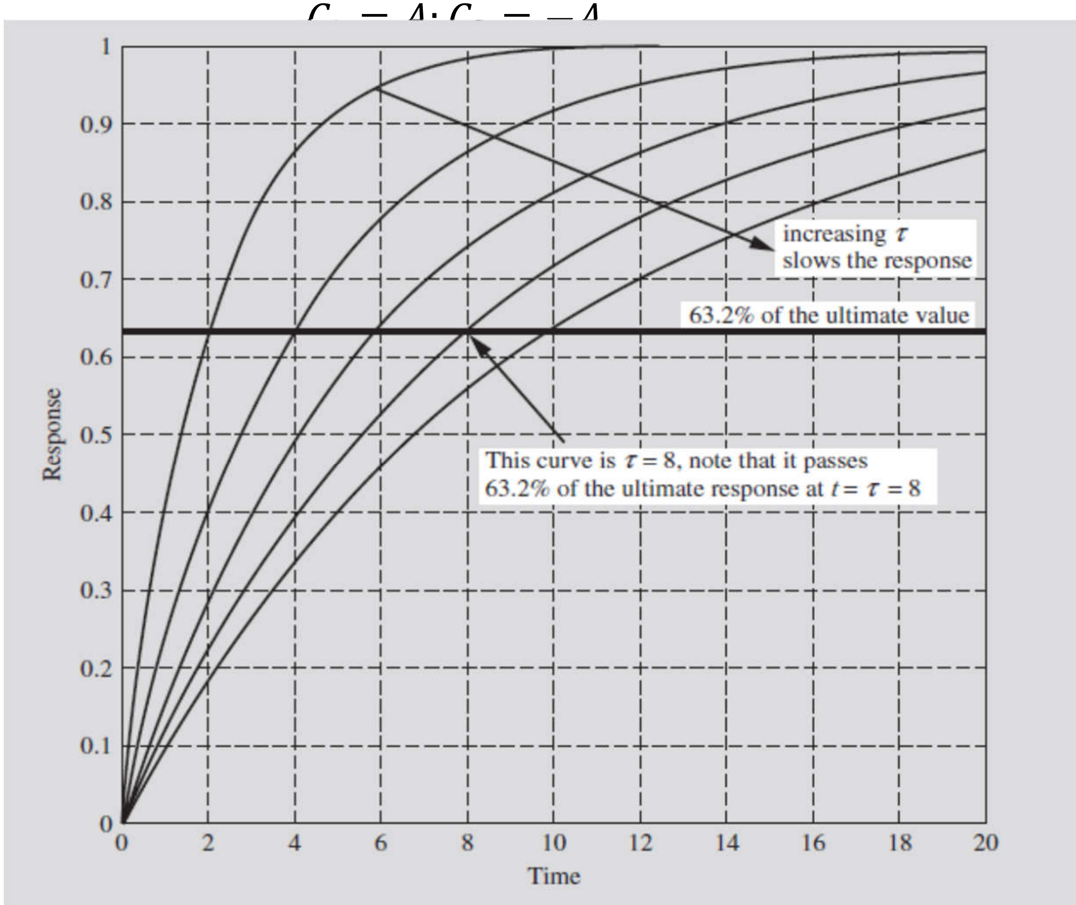
$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{(\tau s+1)}$$

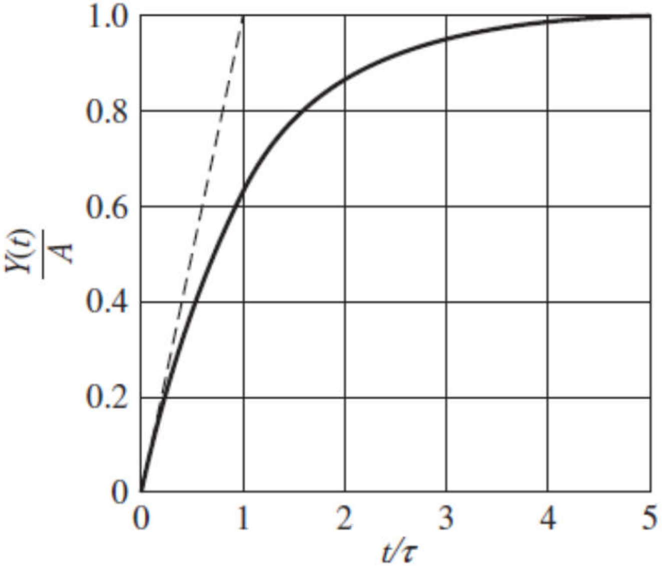
$$Y(s) = \frac{A}{s}\frac{1}{(\tau s+1)}$$

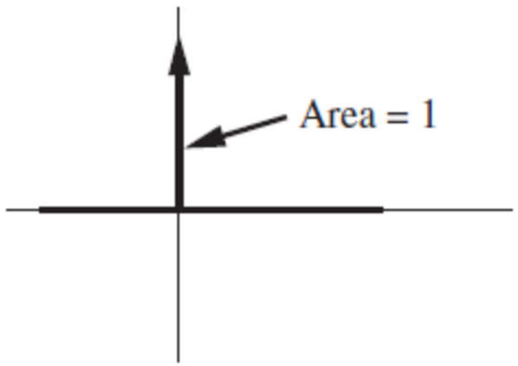
$$Y(s) = \frac{1}{s}\frac{A/\tau}{(s+1/\tau)}$$

$$Y(s) = \frac{C_1}{s} + \frac{C_2}{(s+1/\tau)}$$

$C_1 = A; C_2 = -A$







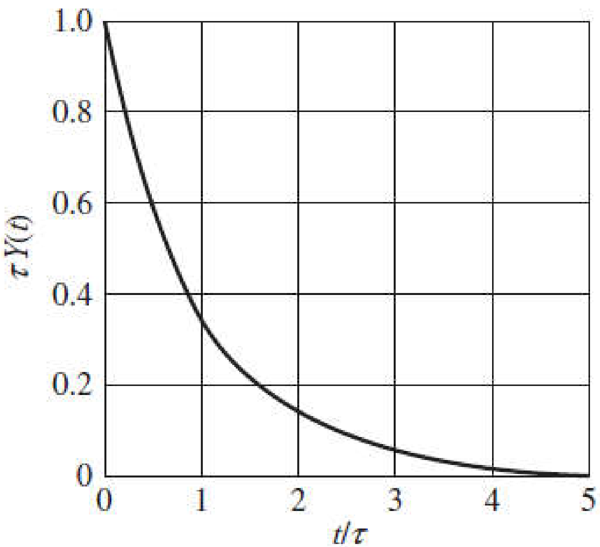
Laplace transform of unit impulse function is $X(s) = 1$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{(\tau s + 1)}$$

$$Y(s) = \frac{1/\tau}{(s + 1/\tau)}$$

$$\tau Y(t) = e^{-t/\tau}$$

1. The value of response Y(t) reaches 63.2 when time elapsed is one time constant
2. Slope of response curve at origin is 1. If initial rate of change in maintained the response of system will be completed in one time constant.



2. Impulse Function

$$X(t) = 0\,; t < 0$$
$$= \frac{1}{h}\;; 0 < t < h; with\ h \to 0$$
$$= 0\,; t > 0$$

3. Ramp Function

$$\boldsymbol{X(t)} = \begin{Bmatrix} \mathbf{0} & \boldsymbol{for\ t < 0} \\ \boldsymbol{t} & \boldsymbol{for\ t > 0} \end{Bmatrix} = \boldsymbol{tu(t)}$$

Forcing function

$$\boldsymbol{X(t)} = bt \qquad \boldsymbol{X(s)} = \frac{b}{s^2}$$

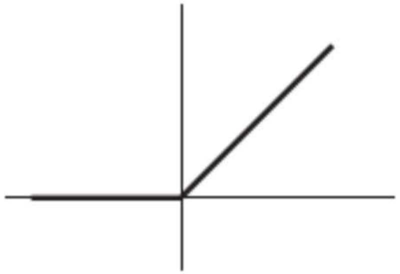
$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{(\tau s + 1)}$$

$$Y(s) = \frac{1}{s^2}\frac{b/\tau}{(s + 1/\tau)}$$

$$\frac{1}{s^2}\frac{b/\tau}{(s + 1/\tau)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s + \frac{1}{\tau}}$$

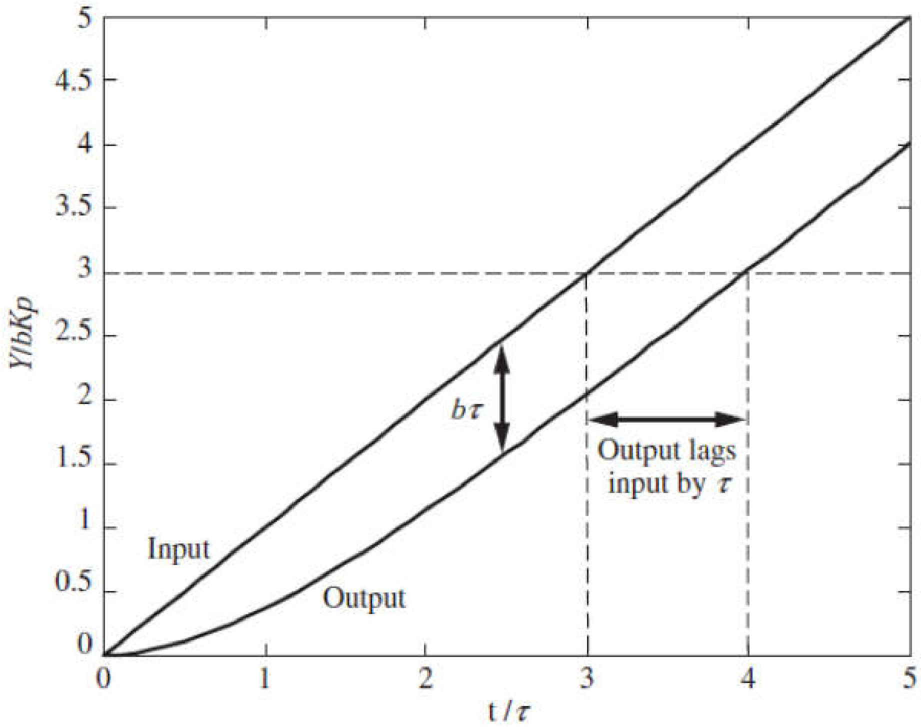
$$A = b; B = -b\tau; C = b\tau$$

$$Y(s) = \frac{b}{s^2} + \frac{-b\tau}{s} + \frac{b\tau}{s + \frac{1}{\tau}}$$



$$Y(t) = bt - b\tau + b\tau e^{-\frac{t}{\tau}}$$

$$Y(t) = bt - b\tau(1 - e^{-\frac{t}{\tau}})$$



Dynamic Error

The difference between input variable $\boldsymbol{X(t)}$ and response at steady state is called as dynamic error

Dynamic error

$$= \boldsymbol{X(t)} - \boldsymbol{Y(t)}$$

$$= bt - b(t - \tau)$$

$$= b\tau$$

Time lag

Time taken by the response to reach the value of input

$$\frac{\boldsymbol{X(t)}}{b} = \frac{\boldsymbol{Y(t)}}{b}$$

$$t_i = t_f - \tau$$

$$t_f - t_i = \tau$$

4. Sinusoidal Function

Consider the case of a thermometer initially in steady state with a temperature bath at temperature x_s At time t = 0 the bath temperature varies according to relationship

$$x = x_S + A \sin \omega t \qquad t > 0$$

Introducing deviation variables

$$X = x - x_s$$

$$X = A \sin \omega t$$

$$X(s) = \frac{A\omega}{s^2 + \omega^2}$$

Transfer function for first order system is

$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{(\tau s + 1)}$$

$$Y(s) = \frac{A\omega}{s^2 + \omega^2} \frac{1/\tau}{(s + 1/\tau)}$$

Taking inverse Laplace transform and solving for Y(t)

$$Y(t) = \frac{A\omega\tau e^{-t/\tau}}{1 + \tau^2\omega^2} - \frac{A\omega\tau}{1 + \tau^2\omega^2} \cos \omega t + \frac{A}{1 + \tau^2\omega^2} sin\omega t$$

Using form of trigonometric identity

$$p \cos B + q \sin B = r \sin (B + \theta)$$

where $\quad r = \sqrt{p^2 + q^2} \quad \tan \theta = \frac{p}{q}$

$$p = -\frac{A\omega\tau}{1 + \tau^2\omega^2} \qquad q = \frac{A}{1 + \tau^2\omega^2}$$

$$r = \frac{A}{\sqrt{1 + \tau^2\omega^2}} \qquad tan\theta = -\omega\tau$$

Applying the identity to Y(t)

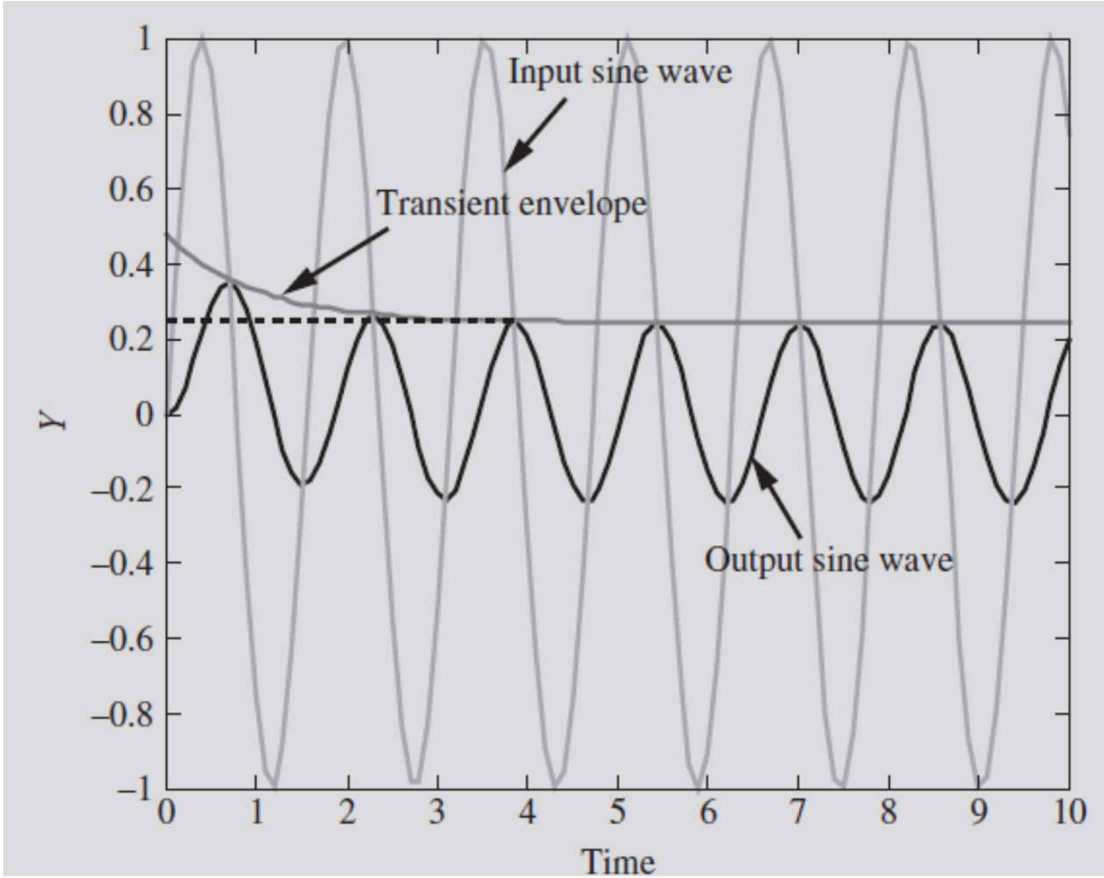
$$Y(t) = \frac{A\omega\tau e^{-t/\tau}}{1 + \tau^2\omega^2} + \frac{A}{\sqrt{1 + \tau^2\omega^2}} \sin(\omega t + \theta)$$

$$Y(t) = \frac{A\omega\tau e^{-t/\tau}}{1+\tau^2\omega^2} + \frac{A}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t + \theta)$$

$$When\ t \to \infty$$

$$Y(t)_{Steady} = \frac{A}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t + \theta)$$

1. The output is a sine wave with a frequency ω equal to that of input signal
2. The ratio of output amplitude to that of input amplitude is called as Amplitude ratio which is $\frac{1}{\sqrt{1+\tau^2\omega^2}}$ less than 1. i.e. the output signal has reduced in strength (attenuated).
3. The output lags the input by an angle of θ. The lag occurs as the sign of θ is always negative.



Problems for Practice

A thermometer has a time constant of 2 seconds. The thermometer reads the temperature which is suddenly increased from 800 to 900 $℃$ *and is mantained at that temeperature.* What is the temperature indicated by thermometer 6 seconds after change the process is made?

$$Y(t) = A(1 - e^{-t/\tau})$$

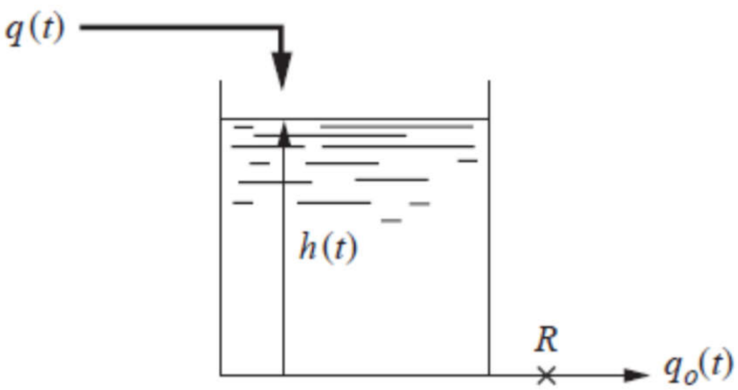
A first order element is subjected to step change of magnitude 50 units. The output from the system is about 15.8 units at time t = 15 seconds. Calculate the time constant of the element.

A first order element is subjected to unit impulse. The output as a function of time is shown below. Calculate the time constant from the data

A temperature alarm may be taken to behave like a first order system with a time constant of 1 min. Due to fire temperature of surrounding go up by 200 $℃$. If an increase of 50 $℃$ is required to activate the alarm, what is the delay before alarm rings?

Time (sec)	0.5	1	2	2.5	3	4	5
Y(t)	0.181	0.163	0.134	0.121	0.11	0.09	0.079

Examples of first order systems



If a time varying volumetric flow q(t) of constant density '**ρ**' enters the tank.
Determine the transfer function that relates head to flow.

Writing a transient mass balance around the tank .

Rate of mass flow in	-	Rate of mass flow out	=	Rate of accumulation of mass in tank
$\rho q(t)$	-	$\rho q_0(t)$	=	$\dfrac{d(\rho A h)}{dt}$

Consider a tank of cross sectional area A, a flow resistance (R) is attached.

Volumetric flow rate $\mathbf{q_0(t)}$ through resistance is related to head by relation

$$q_0(t) = \frac{h}{R}$$

Linear resistance – direct relationship between flow rate and head is known as linear resistance

$$q(t) - q_0(t) = A\frac{dh}{dt} \qquad q(t) - \frac{h}{R} = A\frac{dh}{dt} \quad \textit{\textbf{equation 1}}$$

Introducing deviation variables when the process operates at steady state .

$$q_s(t) - \frac{h_s}{R} = A\frac{dh_s}{dt} \quad \textit{\textbf{equation 2}}$$

$$\boldsymbol{equation\,1\ -equation\,2}$$

$$q - q_s = \frac{(h - h_s)}{R} + A\frac{d(h - h_s)}{dt}$$

$$q - q_s = \mathrm{Q} \quad h - h_s = \mathrm{H}$$

$$Q = \frac{H}{R} + A\frac{dH}{dt}$$

Taking the Laplace transform on both sides

$$Q(s) = \frac{H(s)}{R} + AsH(s)$$

$$\boldsymbol{\frac{H(s)}{Q(s)} = \frac{R}{\tau s + 1}} \qquad \boldsymbol{\tau = \mathrm{AR}}$$

For a step change in inlet flow rate

$$Q(t) = u(t) \qquad Q(s) = \frac{1}{s}$$

$$H(s) = \frac{1}{s}\frac{R}{\tau s + 1}$$

Corresponding change in head in a steady state once the inlet flow rate (disturbance) enters the system is

$$H(t)_{t\to\infty} = \lim_{s\to 0} sH(s)$$

$$\lim_{s\to 0} s\frac{1}{s}\frac{R}{\tau s + 1}$$

$$= R$$

For transfer function relating inlet and outlet flow

$$q_0 = \frac{h}{R} \qquad q_{0s} = \frac{h_s}{R}$$

$$Q_0 = \frac{H}{R} \qquad Q_0 = q_0 - q_{0s}$$

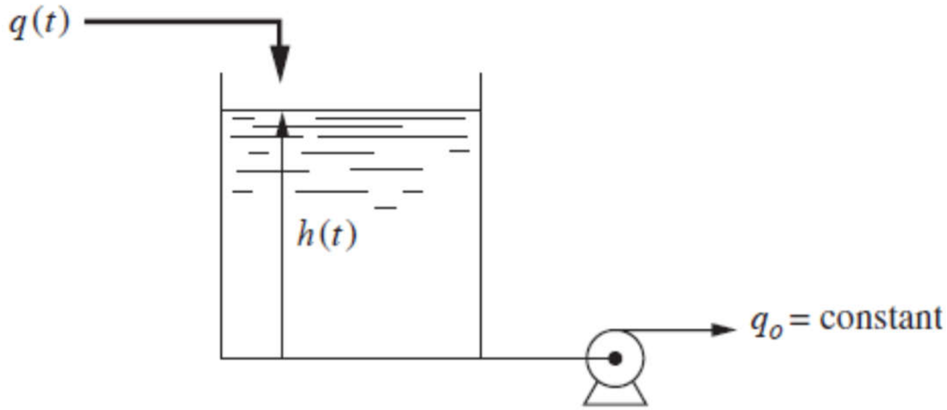
$$Q_0(s) = \frac{H(s)}{R}$$

$$H(s) = Q_0(s)R$$

$$\frac{Q_0(s)R}{Q(s)} = \frac{R}{\tau s + 1}$$

$$\boldsymbol{\frac{Q_0(s)}{Q(s)} = \frac{1}{\tau s + 1}}$$

Liquid level process with constant outlet



$$q(t) - q_0 = A\frac{dh}{dt}\,\boldsymbol{equation\ 1}$$

At steady state

$$q_s - q_0 = A\frac{dh_s}{dt}\,\boldsymbol{equation\ 2}$$

$$\boldsymbol{equation\ 1\ - equation\ 2}$$

$$q - q_s = A\frac{d(h - h_s)}{dt}$$

$$q - q_s = \mathrm{Q} \quad h - h_s = \mathrm{H}$$

$$Q = A\frac{dH}{dt}$$

Taking Laplace transform

$$Q(s) = AsH(s)$$

$$\frac{H(s)}{Q(s)} = \frac{1}{As}$$

$$H(s) = \frac{1}{As}Q(s)$$

On integration

$$h(t) = h_s + \frac{1}{A}\int_0^t Q(t)dt$$

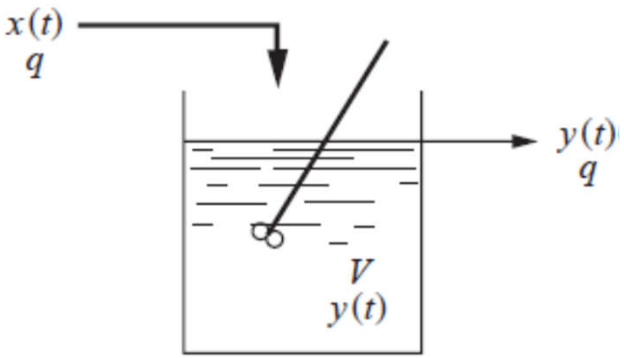
For a step change in flow rate

$$Q(t) = u(t)$$

$$h(t) = h_s + \frac{t}{A}$$

Step response is given by ramp function which grows with out limit - non regulatory system.

Mixing Process



$q = volumetric\ flow\ rate\ of\ solution$

$x(t) = concentration\ of\ salt\ in\ entering\ solution$

(mass of salt/volume)

$y(t) = concentration\ of\ salt\ in\ tank\ or\ in\ leaving\ solution$

(mass of salt/volume)

$Determine\ transfer\ function\ relating\ outlet\ to\ inlet\ concentration$

$Assuming\ constant\ density\ writing\ mass\ balance\ for\ salt$

flow rate of salt in	-	flow rate of salt out	=	Rate of accumulation of salt in tank

$$qx \quad - \quad qy \quad = \quad V\frac{dy}{dt} \qquad equation\ 1$$

$At\ steady\ state$

$$qx_s \quad - \quad qy_s \quad = \quad V\frac{dy_s}{dt} \qquad equation\ 2$$

$equation\ 1\ - equation\ 2$

$$q(x - x_s) - q(y - y_s) = V\frac{d(y - y_s)}{dt}$$

$Introducing\ deviation\ variables$

$$X = x - x_s \qquad Y = y - y_s$$

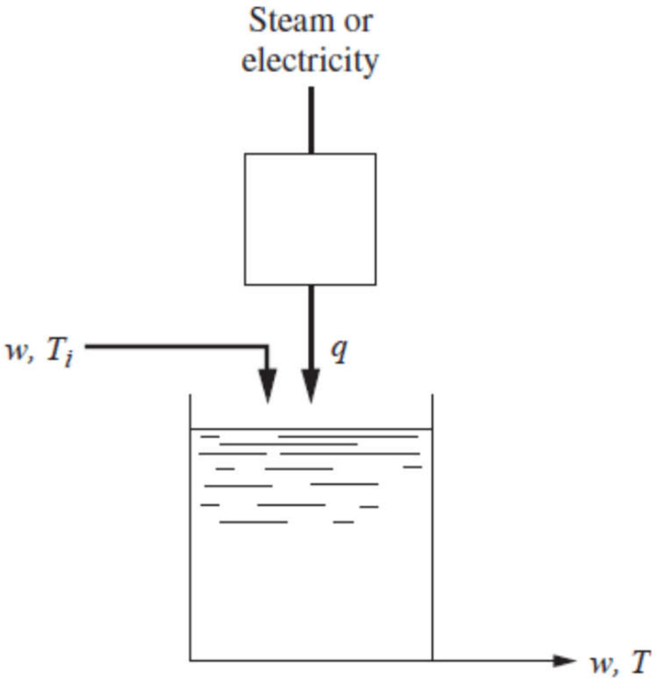
$$qX - qY = V\frac{dY}{dt}$$

$$qX(s) - qY(s) = VsY(s)$$

$$qX(s) = Y(s)(q + Vs)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{(1 + \frac{V}{q}s)} = \frac{1}{1 + \tau s} \qquad \tau = \frac{V}{q}$$

Heating Process



Considering a transient energy balance around the tank

Rate of Energy flow into tank	-	Rate of energy flow out of tank	+	Rate of energy flow from heater	=	Rate of accumulation of energy in tank

$$wC(T_i - T_{ref}) - wC(T - T_{ref}) + q = \rho VC\frac{dT}{dt} \; Eqn\ 1$$

At steady state

$$wC(T_{is} - T_{ref}) - wC(T_s - T_{ref}) + q_s = \rho VC\frac{dT_s}{dt} \; Eqn\ 2$$

$$Eqn\ 1 - Eqn\ 2$$

$$wC(T_i - T_{is}) - wC(T - T_s) + q - q_s = \rho VC\frac{d(T - T_s)}{dt}$$

At steady state $T_i = T_{is}$ ***Introducing deviation variables***

$$T' = T - T_s \qquad Q = q - q_s$$

$$-wCT' + Q = \rho VC\frac{dT'}{dt}$$

Taking Laplace trtansforms

$$-wCT'(s) + Q(s) = \rho VCsT'(s)$$

$$\frac{T'(s)}{Q(s)} = \frac{1/wC}{(\rho V/w)s + 1} = \frac{K}{\tau s + 1}$$

Linearization

Examples of physical system has been considered were linear

Linear resistance – direct relationship between flow rate and head is known as linear resistance

$$q_0(t) = \frac{h}{R}$$

Systems of practical importance are generally non – linear.

$$q_0(t) = Ch^{1/2}$$

Writing a transient mass balance around the tank .

$$q(t) - q_0(t) = A\frac{dh}{dt}$$

$$q(t) - Ch^{1/2} = A\frac{dh}{dt}$$

Expanding $q_0(h)$ around steady state value h_s *by means of taylor series expansion*

$$q_o = q_o(h_s) + q_o'(h_s)(h - h_s) + \frac{q_o''(h_s)(h - h_s)^2}{2} + \cdots$$

Considering linear term in Taylor series expansion

$$q_o \cong q_o(h_s) + q_o'(h_s)(h - h_s)$$

Taking the derivative of $q_0(t)$ around steady state value h_s

$$q_o'(h_s) = \frac{1}{2}Ch_s^{-1/2}$$

$$q_0 = q_{0s} + \frac{1}{2}Ch_s^{-\frac{1}{2}} \ (h - h_s)$$

$$\frac{1}{R_1} = \frac{1}{2}Ch_s^{-\frac{1}{2}}$$

$$q_0 = q_{0s} + \frac{1}{R_1} \ (h - h_s)$$

$$q(t) - q_0(t) = A\frac{dh}{dt}$$

$$q_0 = q_{0s} + \frac{1}{R_1}\,(h - h_s)$$

$$q(t) - q_0(t) = A\frac{dh}{dt}$$

$$q - q_s - \frac{1}{R_1}\,(h - h_s) = A\frac{dh}{dt}$$

$$q - q_s = \frac{1}{R_1}\,(h - h_s) + A\frac{dh}{dt}$$

$$Q = \frac{H}{R_1} + A\frac{dH}{dt}$$

$Q = q - q_s$ and $H = h - h_s$

$$\frac{H(s)}{Q(s)} = \frac{R_1}{\tau s + 1}$$

$$R_1 = \frac{2h_s^{1/2}}{C} \qquad \tau = R_1 A$$